PhD. Defense

Theory of Sequences Tailored for Program Verification

Théorie des séquences adaptée à la vérification des programmes

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Under the supervision of **François BOBOT**[2] and **Guillaume BURY**[1]

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Software is everywhere

Personal information

Energy

Aerospace

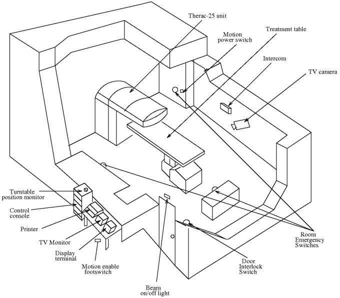
Healthcare

And it can be faulty (bugged)

Ariane 5 *

(Arithmetic overflow)

Loss of over US$370 million



Therac-25 **

(Race condition)

Deaths and injuries of patients

More examples:
`https://en.wikipedia.org/wiki/List_of_software_bugs`

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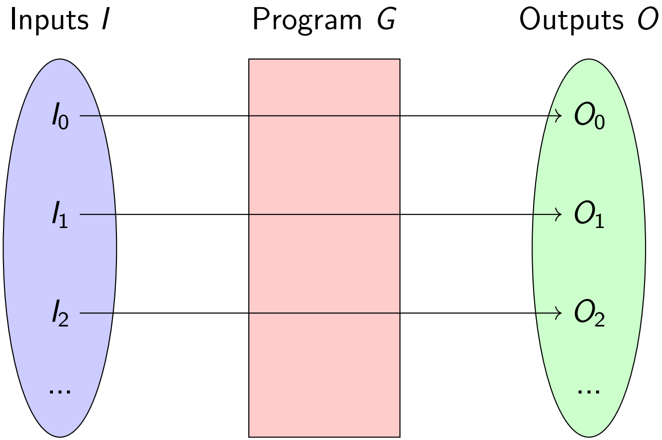
*Photo: ©ESA

**Figure from: "Medical Devices: The Therac-25" by Nancy G. Leveson

Behind the software, there are programs.



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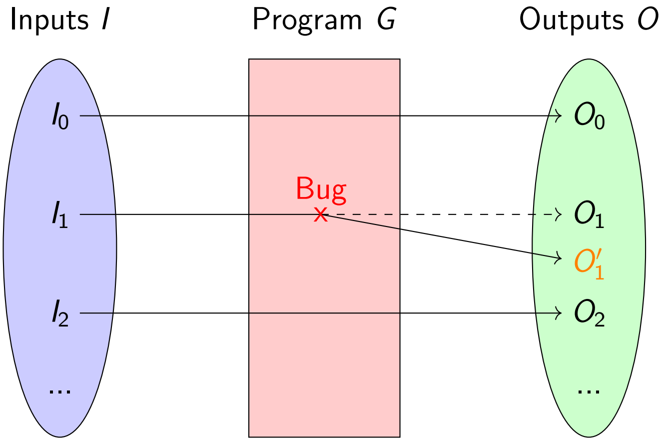

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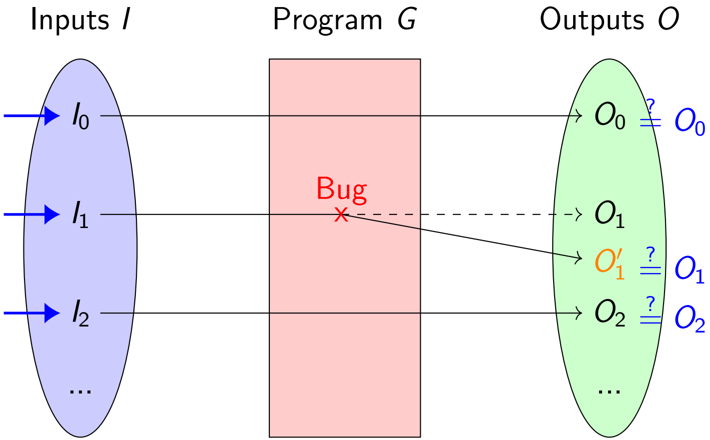
Testing



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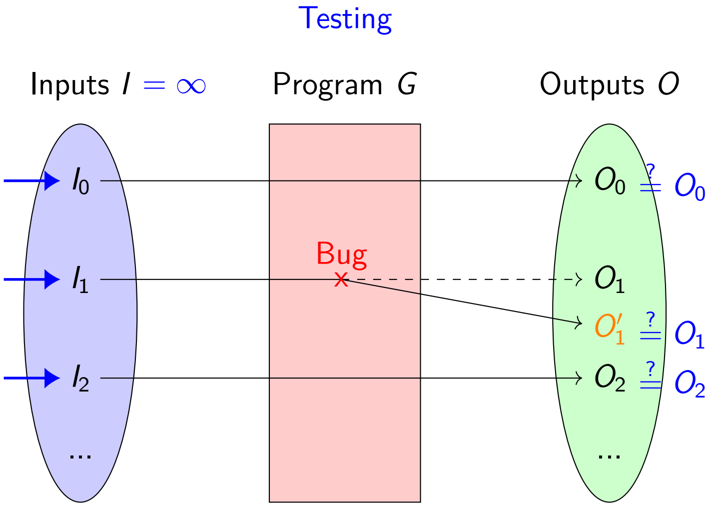

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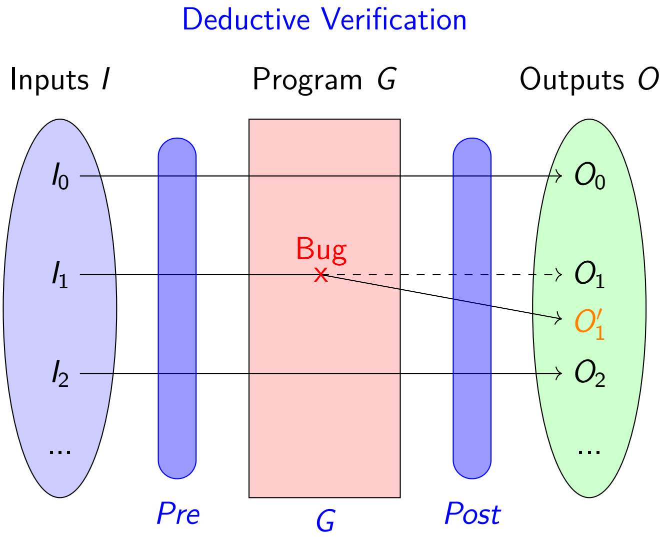

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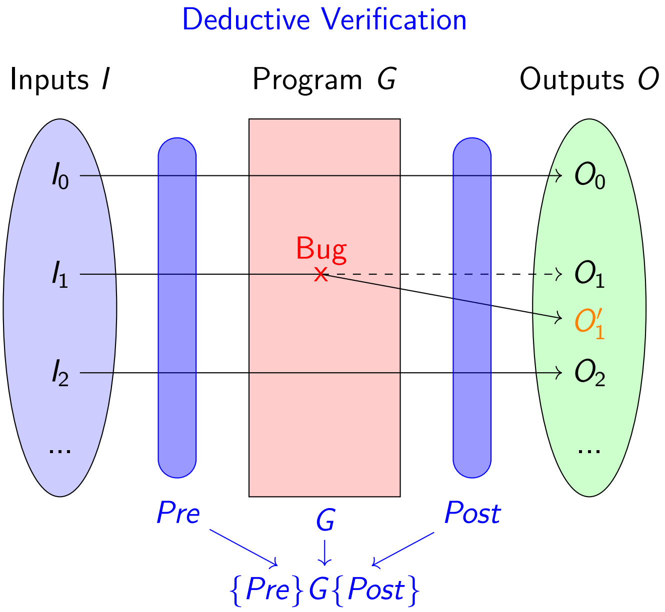

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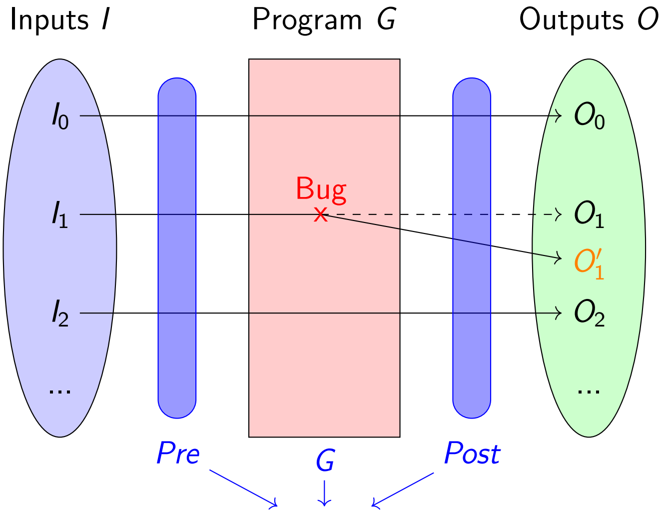
Deductive Verification



$\{Pre\}G\{Post\} \Rightarrow \Psi$ (proven?)

Avoiding Bugs in the Real World

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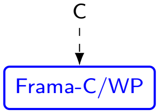

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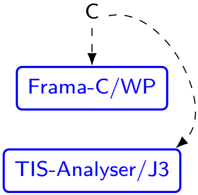

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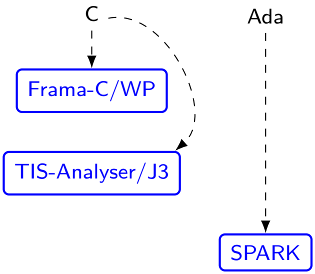

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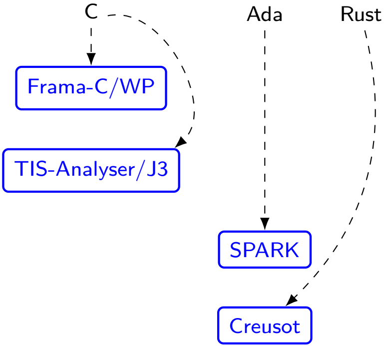

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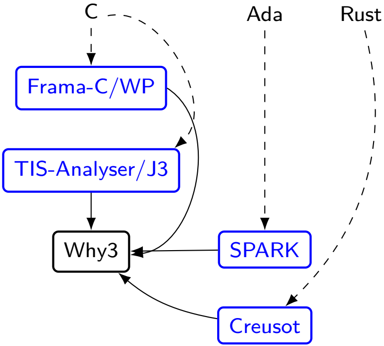

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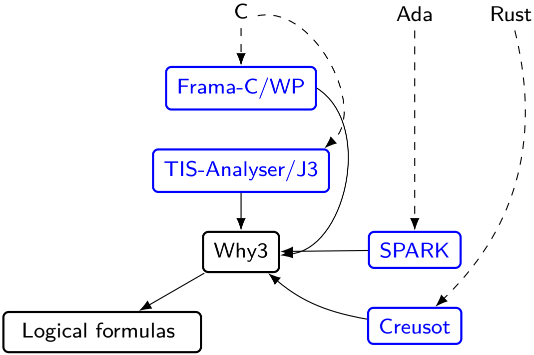

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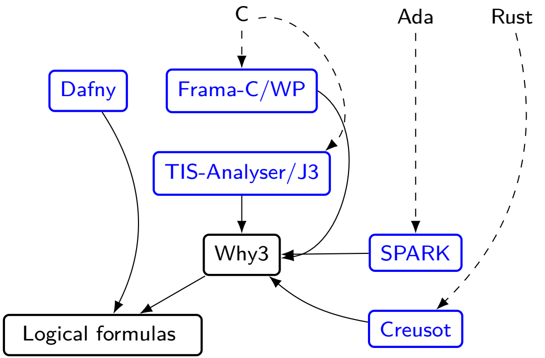

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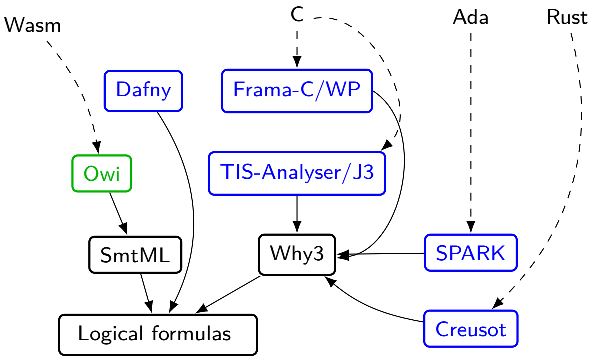

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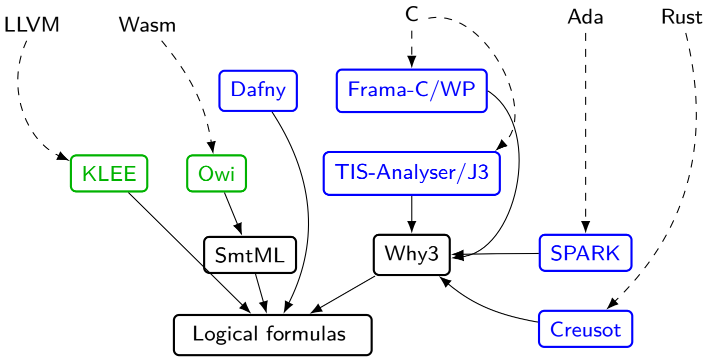

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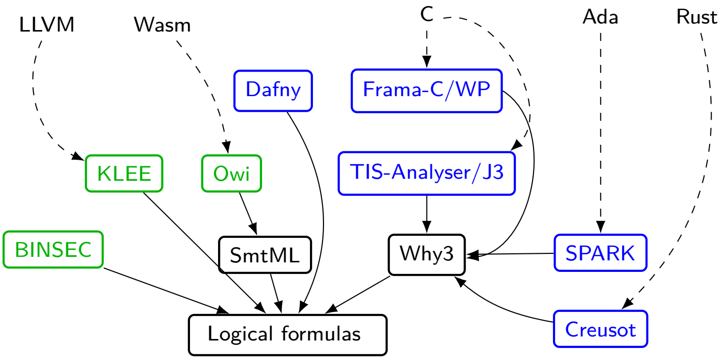

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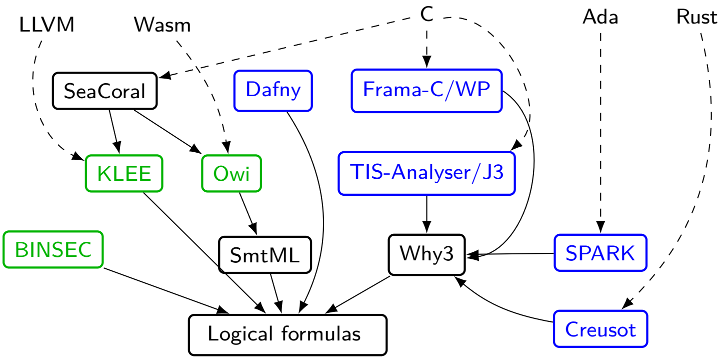

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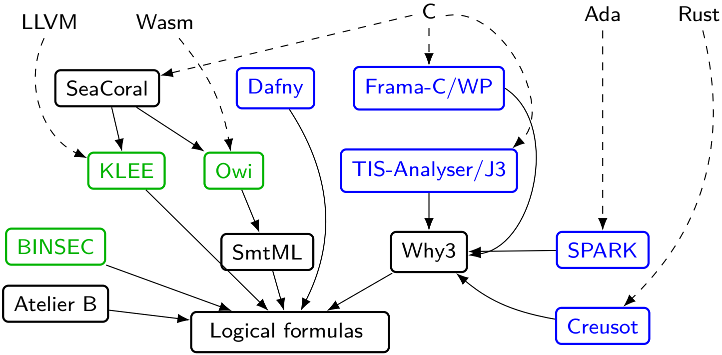

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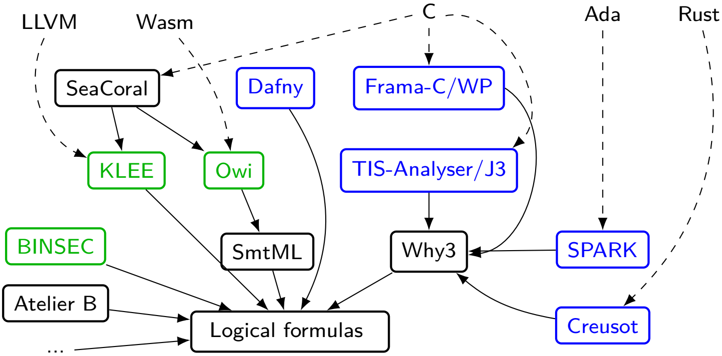

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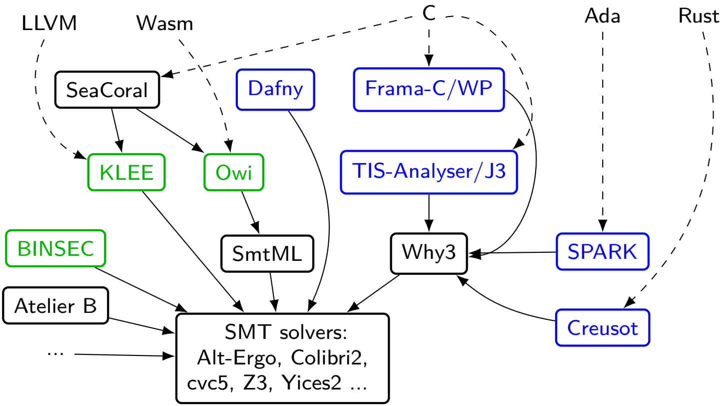

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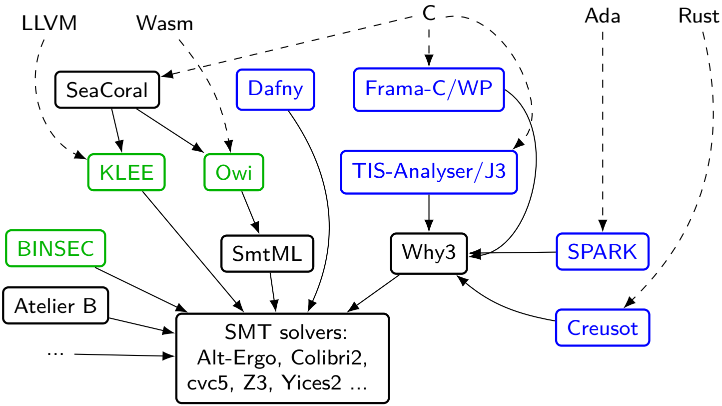

In the industry:

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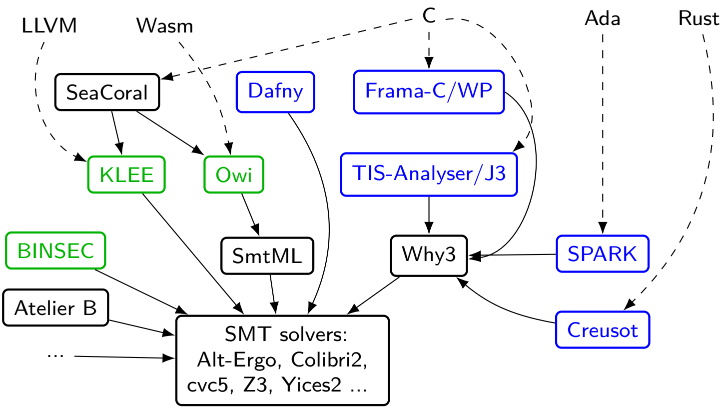

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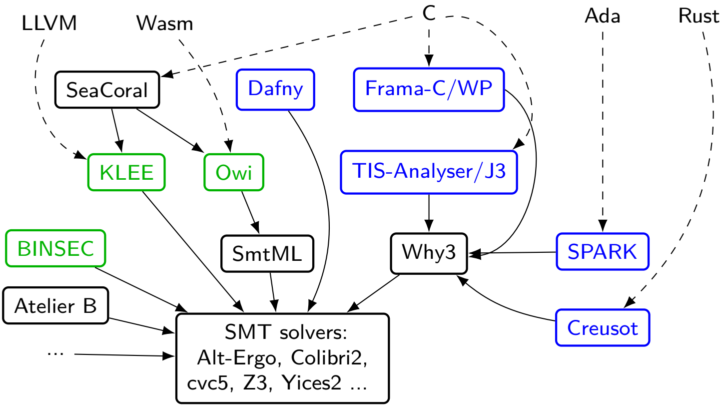

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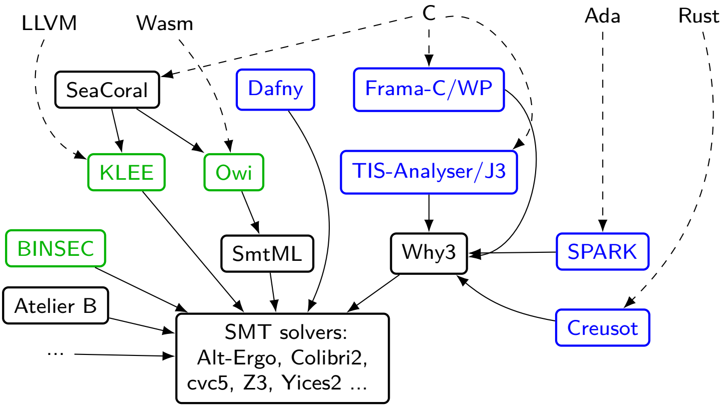

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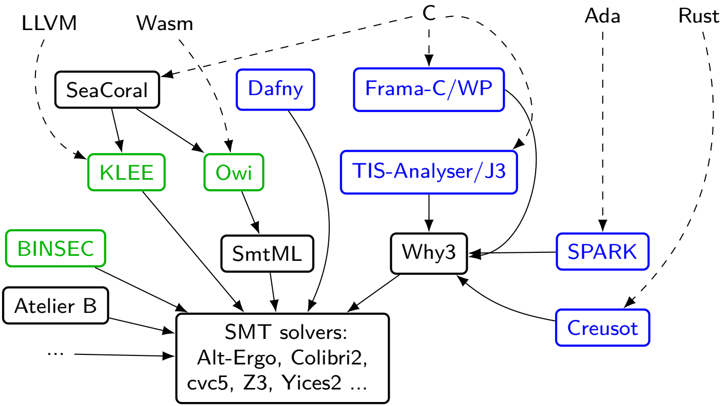

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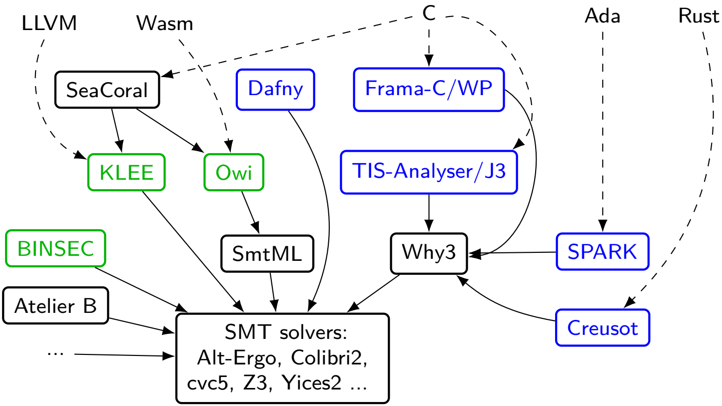

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What is SMT (Satisfiability Modulo Theories)?

Definition

Boolean Satisfiability (Propositional Logic)

$+$

Built-in First-Order Logic Theories

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Yes, therefore F is **satisfiable**.

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No, therefore F is **unsatisfiable**. Inversely $\neg F$ is **valid**.

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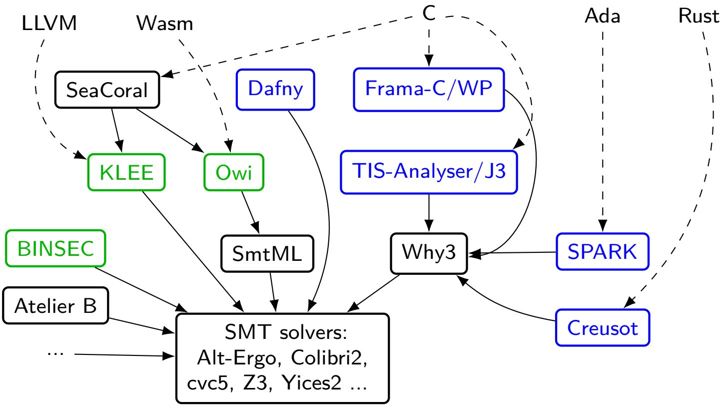

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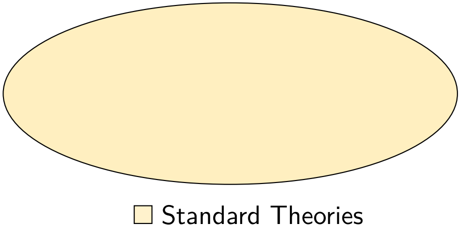

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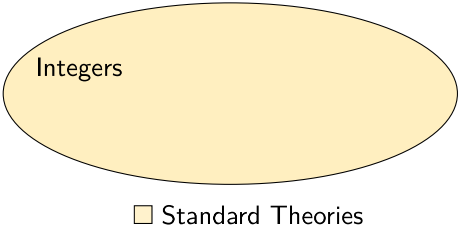

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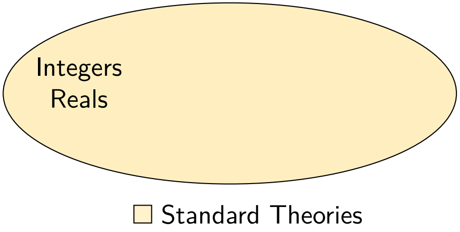

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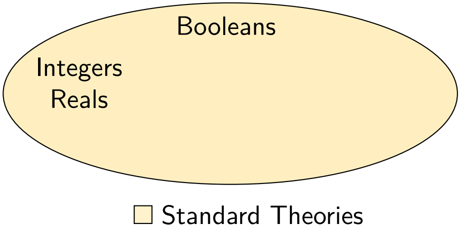

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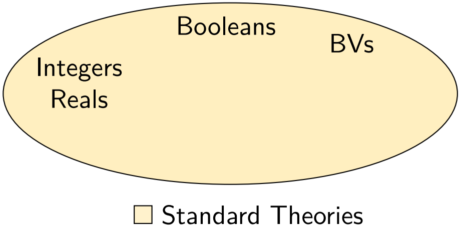

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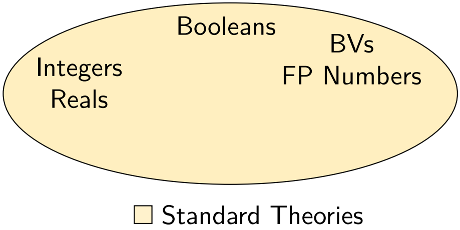

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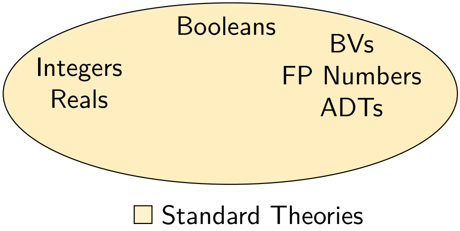

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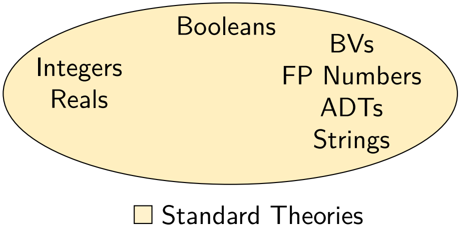

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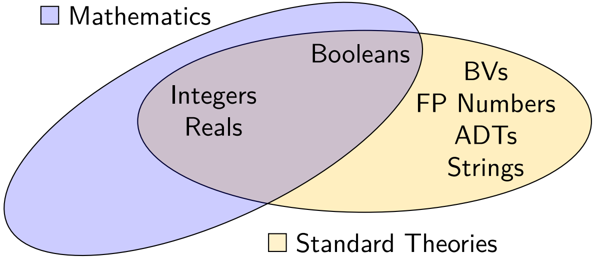

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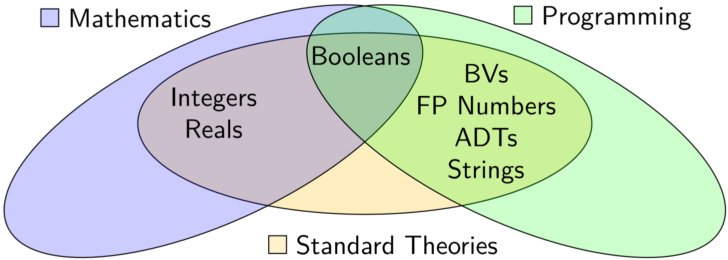

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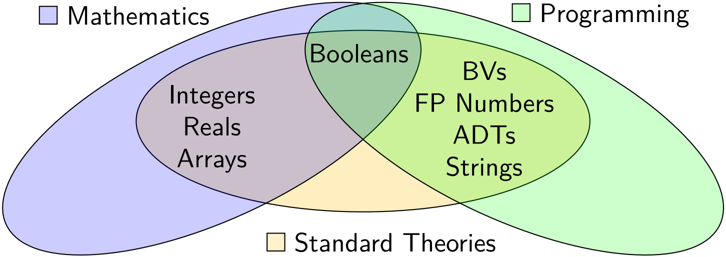

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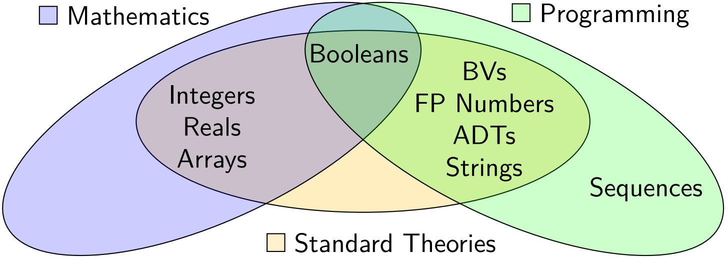

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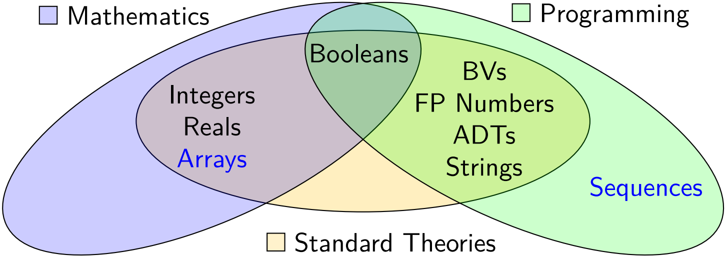

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Sequences are more suitable as they are semantically closer.

In this thesis: A different theory of Sequences

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An n-indexed sequence (or n-sequence) s is a sequence that is indexed from a first index f_s to a last index l_s.

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Outline

1. The SMT theory of n-Indexed Sequences
2. Reasoning over n-Indexed Sequences
3. Implementation
4. Experimental Evaluation
5. Conclusion

Outline

Semantics of the theory of n-Indexed Sequences I

- f_s and l_s: the first and last index of s.

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Empty n-indexed sequence

An n-indexed sequence s is said to be empty if $\mathsf{l}_s < \mathsf{f}_s$. Two empty n-indexed sequences a and b are equal only if $\mathsf{f}_a = \mathsf{f}_b$ and $\mathsf{l}_a = \mathsf{l}_b$.

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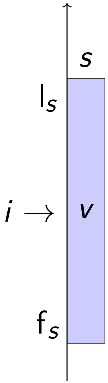

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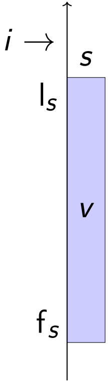

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- relocate(s, f): a copy of s relocated to the index f.

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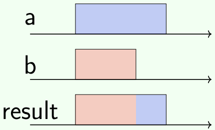

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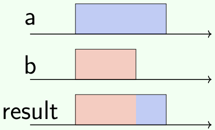

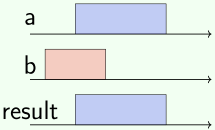

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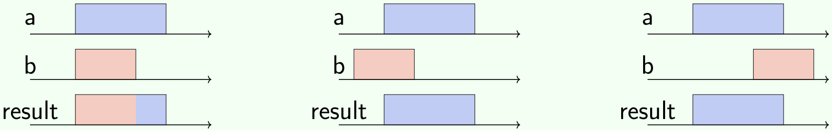

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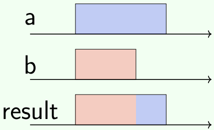
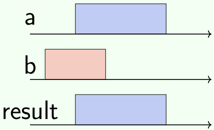
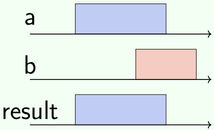
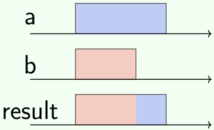
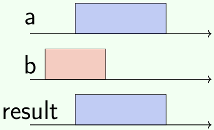


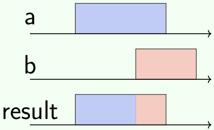




In the theory of sequences (in cvc5) update(a, i, b):







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To read more on SMT theory design and semantic choices:

- **"On SMT Theory Design: The Case of Sequences"**
  Hichem Rami Ait-El-Hara, François Bobot and Guillaume Bury. LPAR 2024

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Extensionality

The theory of n-indexed sequences is extensional. Therefore, given two n-indexed sequences a and b:

$$\begin{gathered}(a = b) \equiv \\ (\mathsf{f}_a = \mathsf{f}_b \wedge \mathsf{l}_a = \mathsf{l}_b \wedge \\ \forall i : \mathsf{Int}, \mathsf{f}_a \leq i \leq \mathsf{l}_a \rightarrow \mathsf{get}(a, i) = \mathsf{get}(b, i))\end{gathered}$$

Outline

Axiomatization/Encoding of n-Sequences

Axiomatization (with arrays):

- Most operations need to be axiomatized.
- Introduces too many quantified formulas.

Encoding using Sequences and Algebraic Data Types:

- Avoids using as many quantifiers.
- Depends on two other theories (Sequences and ADTs).
- Differences in the semantics make the definitions complex.

Reasoning with Sequences and Algebraic Data Types I

n-Indexed sequences are defined as a record:

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Except $\mathsf{const}(f, l, v)$, which has no counterpart in the theory of sequences:

- It can be axiomatized:

$$n = \mathsf{const}(f, l, v) \iff \mathsf{f}_n = f \wedge \mathsf{l}_n = l \wedge \\ \forall i.\ f \leq i \leq l \implies \mathsf{get}(n, i) = v$$

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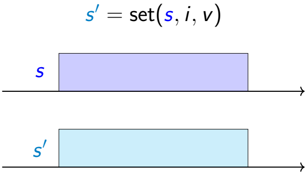

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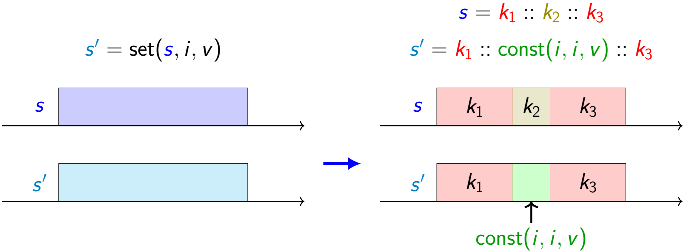

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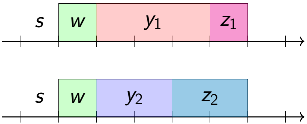

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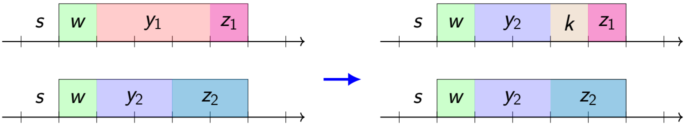

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To read more on NS-BASE and NS-EXT:

- **"An SMT Theory for n-Indexed Sequences"**
  Hichem Rami Ait-El-Hara, François Bobot, and Guillaume Bury. SMT 2024
- **"Reasoning over n-indexed sequences in SMT"**
  Hichem Rami Ait-El-Hara, François Bobot, and Guillaume Bury. Acta Informatica 62.3 (Aug. 2025)

Calculi Summary: NS-BASE and NS-EXT

Operations	NS-BASE	NS-EXT
get set	String reasoning	Array reasoning
concat slice update ...	String reasoning	String reasoning

Limitations:

- Eager normalization is often costly and sometimes unnecessary.

Alternative:

- A new calculus that lazily reasons over slices.

Outline

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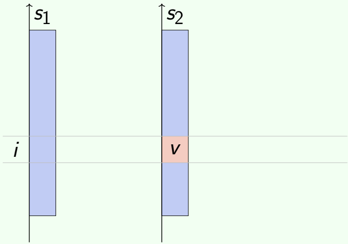

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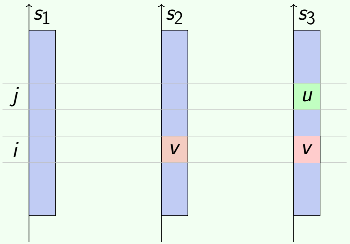
Illustration

Given $s_3 = \mathrm{set}(s_2, j, u)$,
With $\mathrm{f}_{s_2} \leq j \leq \mathrm{l}_{s_2}$:

- $\mathrm{f}_{s_2} = \mathrm{f}_{s_3} \wedge \mathrm{l}_{s_2} = \mathrm{l}_{s_3}$
- $s_1 \overset{\{i,j\}}{\leftrightarrow} s_3$



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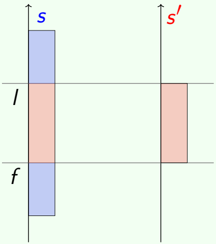

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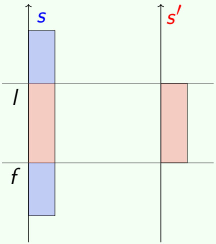

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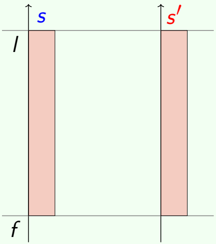

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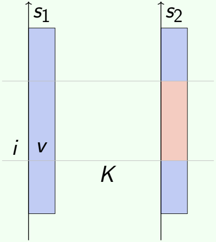

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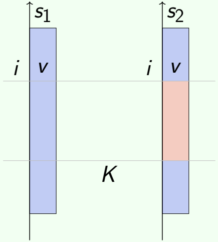

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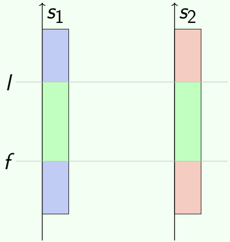

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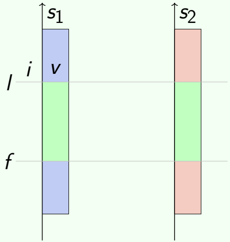

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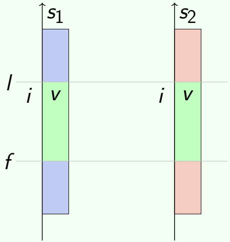
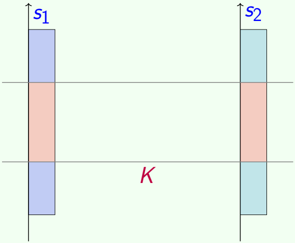


Extensionality with NS-ShS

Ext-ShS $s_1 \overset{K}{\leftrightarrow} s_2$

Illustration



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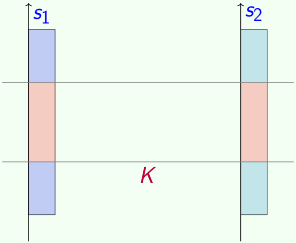

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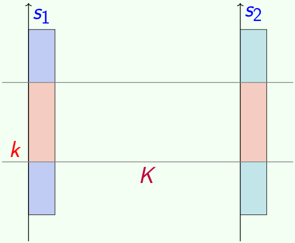

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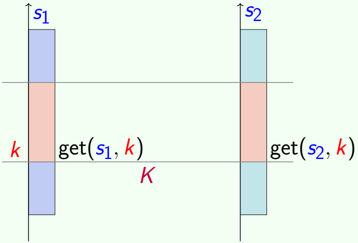

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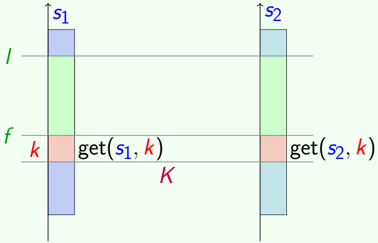

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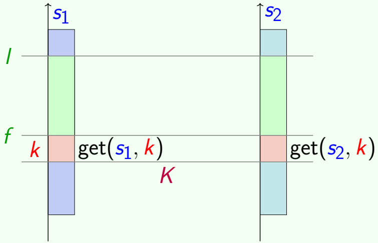

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Outline

Reasoning over relocation 1

Definition (Equivalence modulo relocation)

Given s_1 and s_2 two n-indexed sequences, equivalence modulo relocation is denoted with the relation $s_1 =_{reloc} s_2$, such that:

$$s_1 =_{reloc} s_2 \equiv$$
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$$\forall i : Int, \mathsf{f}_{s_1} \leq i \leq \mathsf{l}_{s_1} \Rightarrow \mathrm{get}(s_1, i) = \mathrm{get}(s_2, i - \mathsf{f}_{s_1} + \mathsf{f}_{s_2})$$

$$\text{Reloc-Bounds}\ \frac{s' = \mathrm{relocate}(s, i)}{\begin{array}{c} i = \mathsf{f}_s \land s' = s \qquad || \\ i \neq \mathsf{f}_s \land \mathsf{f}_{s'} = i \land \mathsf{l}_{s'} = i + \mathsf{l}_s - \mathsf{f}_s \land \end{array}}$$

Reasoning over relocation 1

Definition (Equivalence modulo relocation)

Given s_1 and s_2 two n-indexed sequences, equivalence modulo relocation is denoted with the relation $s_1 =_{reloc} s_2$, such that:

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This reasoning is used for all the three calculi: NS-BASE, NS-EXT and NS-ShS.

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- Applying NS-Comp-Reloc and Get-Reloc eagerly can be **costly**.

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- Applying NS-Comp-Reloc and Get-Reloc eagerly can be **costly**.
- Our extension to the union-find data structure helps mitigate that.

Outline

Implementation context

NS-Base, NS-Ext and NS-ShS were implemented in Colibri2:

- A reimplementation in OCaml of the COLIBRI CP solver.
- A CP solver used to reason over SMT problems.
- That does not use a SAT solver or clause learning.
- Compensates with (abstract) domains, propagations and scheduling.

Outline

Equivalence modulo relocation I

The equivalence modulo relocation relation is represented using a **labeled union-find**.

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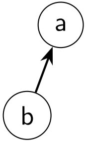
union(a, b)



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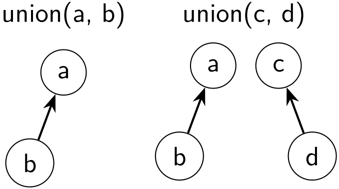

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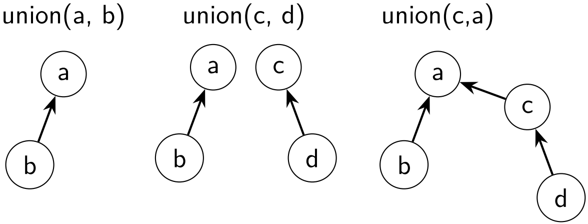

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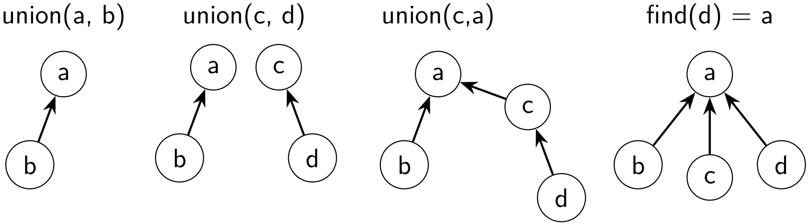

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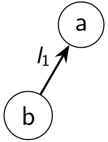
add_relation(b, l_1, a)



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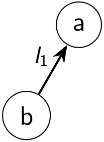
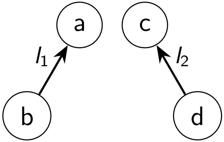


add_relation(d, l_2, c)



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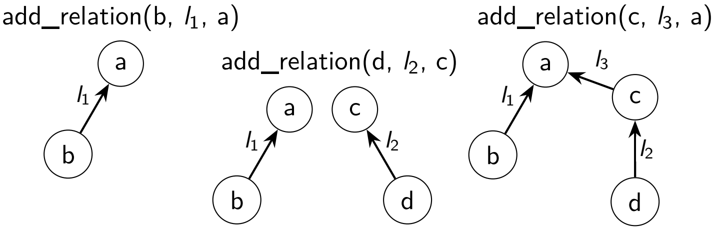

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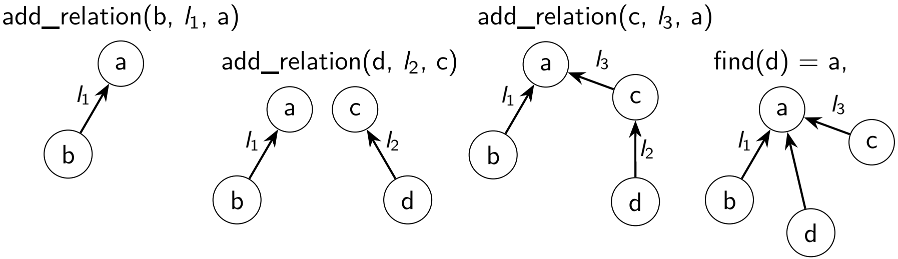

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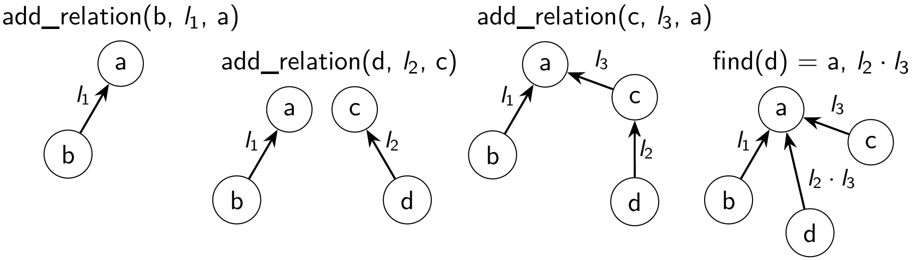

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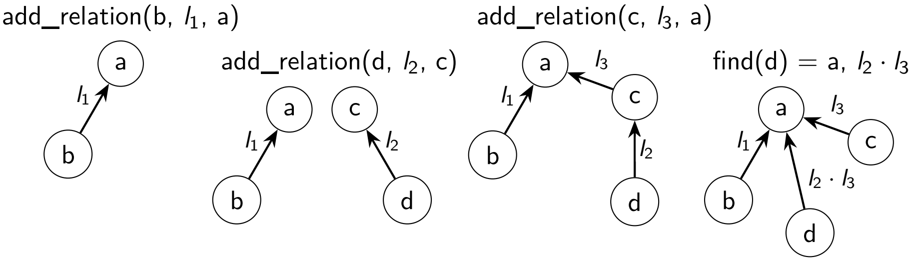

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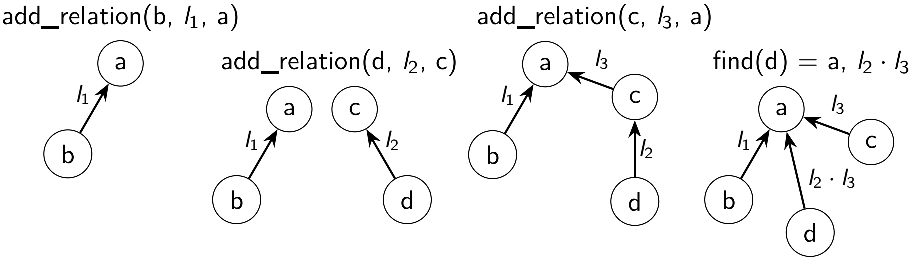

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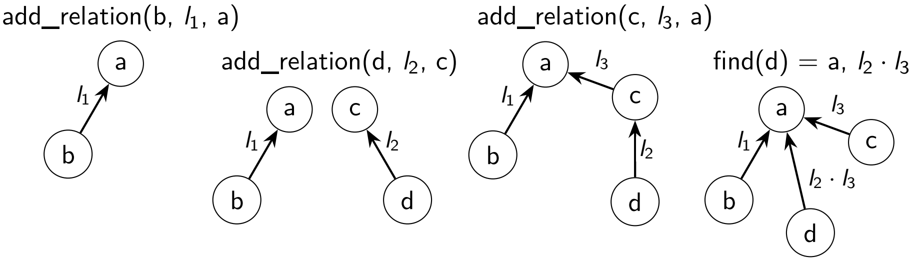

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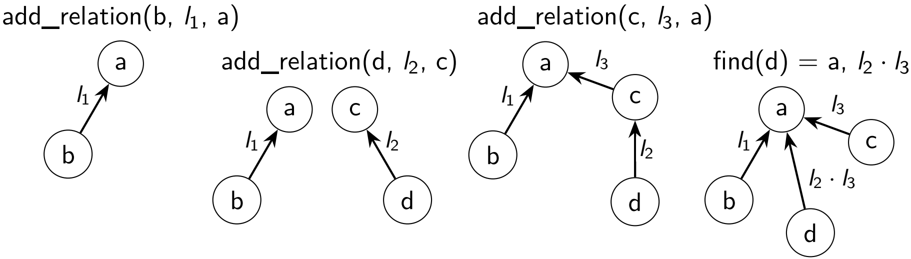

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Forming a **group** with the labels.

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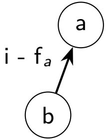

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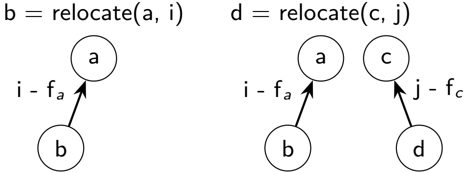

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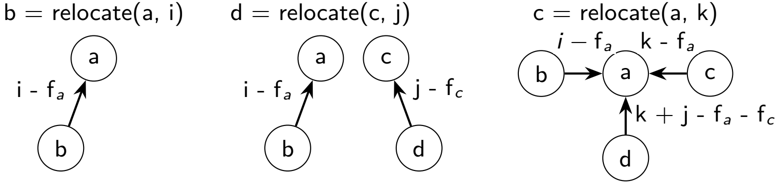

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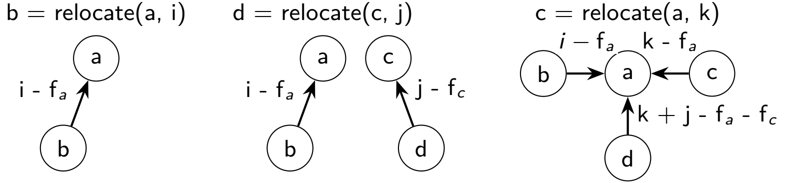

In the implementation, it also holds a domain:

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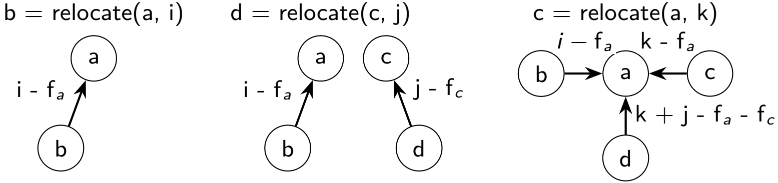

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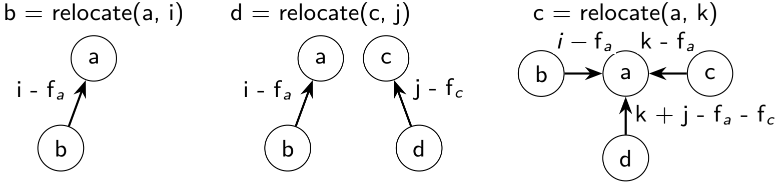

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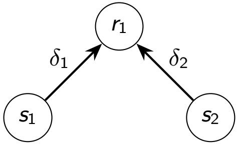
Outline

Constraint factorization



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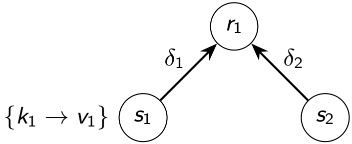

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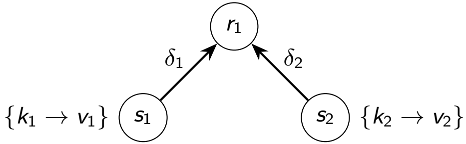

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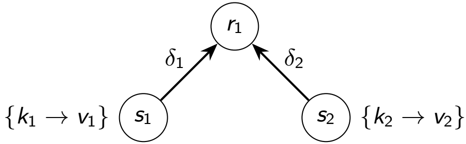

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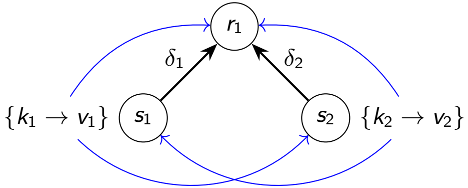

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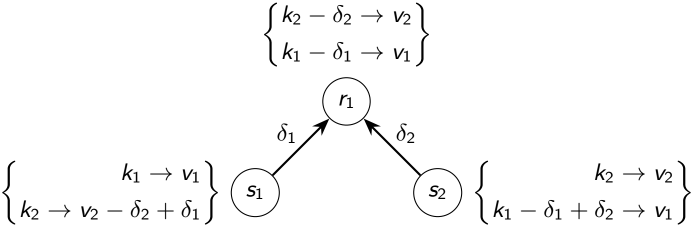

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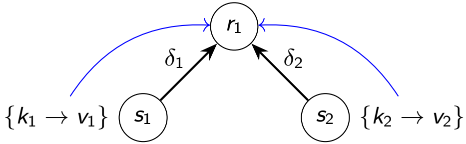

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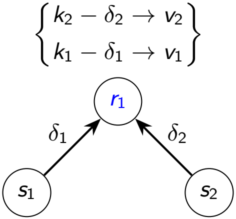

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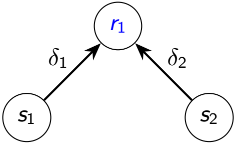
$$\begin{Bmatrix} k_2 - \delta_2 \rightarrow v_2 \\ k_1 - \delta_1 \rightarrow v_1 \end{Bmatrix}$$



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Also applies to NS-Comp-Reloc:

$$\text{NS-Comp-Reloc}\,\frac{s = k_1 :: k_2 :: ... :: k_n \qquad s =_{reloc} r}{\begin{array}{c} r = \text{relocate}(k_1, \mathrm{f}_r) :: \\ \text{relocate}(k_2, \mathrm{f}_{k_2} - \mathrm{f}_s + \mathrm{f}_r) :: ... :: \\ \text{relocate}(k_n, \mathrm{f}_{k_n} - \mathrm{f}_s + \mathrm{f}_r) \end{array}}$$

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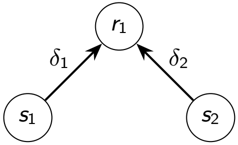

To read more on how it is used in arithmetic reasoning:

- **"Relational Abstractions Based on Labeled Union-Find"**
  Dorian Lesbre, Matthieu Lemerre, Hichem Rami Ait-El-Hara, and François Bobot. PLDI 2025

Outline

Encoding sequences over n-Indexed Sequences

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- seq.empty: Represented by a special constant symbol ϵ, an empty n-sequence with $\mathsf{f}_\epsilon = 0$ and $\mathsf{l}_\epsilon = -1$.
- seq.$++(s_1, s_2, s_3, \ldots, s_n)$:

$$\begin{gathered}\mathrm{let}\big(c_1, \mathrm{concat}(s_1, \mathrm{relocate}(s_2, \mathsf{l}_{s_1} + 1)),\\ \mathrm{let}(c_2, \mathrm{concat}(c_1, \mathrm{relocate}(s_3, \mathsf{l}_{c_1} + 1)),\\ \ldots\\ \mathrm{concat}(c_{n-2}, \mathrm{relocate}(s_n, \mathsf{l}_{c_{n-2}} + 1)))))\big)\end{gathered}$$

Outline

Experimental evaluation: context

- The experimentation was done on quantifier free sequence and n-sequence benchmarks, containing only sequence and n-sequence operations.
- The experimentation compares implementations of NS-BASE, NS-EXT and NS-ShS in Colibri2 with:
  - Sequence support in cvc5 and Z3.
  - Support for n-sequences encoded with ADTs and Sequences in cvc5 and Z3.

Experimental evaluation: UNSAT Seq

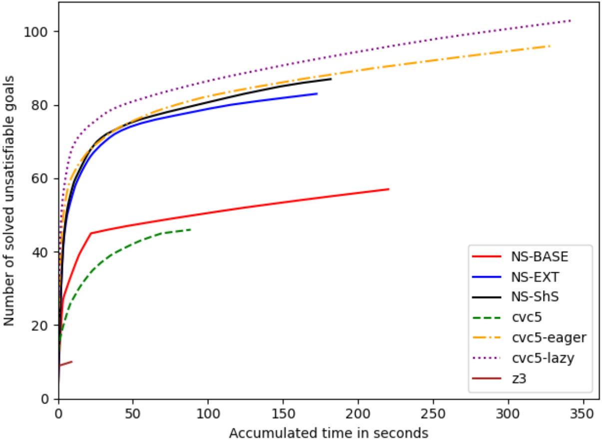

Figure: Number of solved goals by accumulated time (in seconds) on unsatisfiable quantifier-free Sequence benchmarks.

Experimental evaluation: SAT Seq

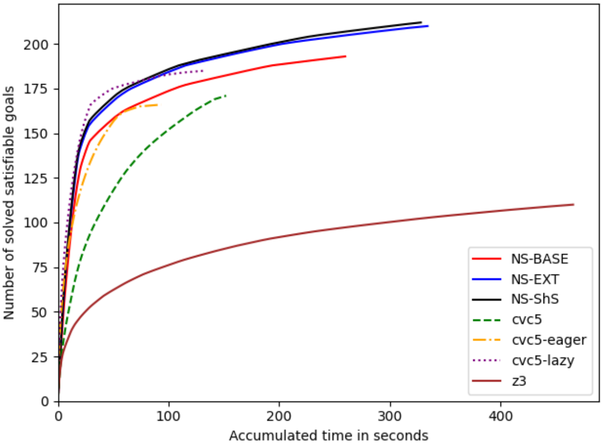

Figure: Number of solved goals by accumulated time (in seconds) on satisfiable quantifier-free Sequence benchmarks.

Experimental evaluation: UNSAT NSeq

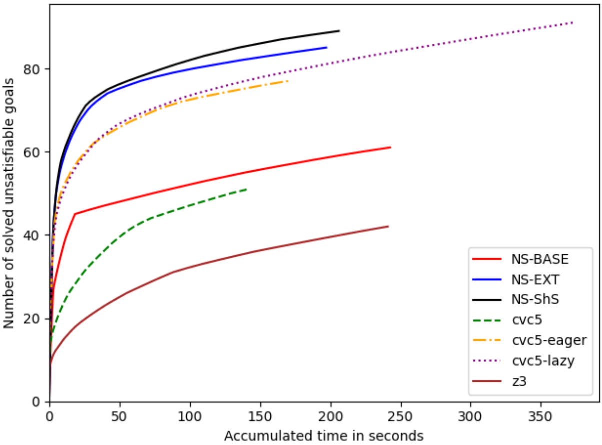

Figure: Number of solved goals by accumulated time (in seconds) on unsatisfiable quantifier-free *n*-Indexed Sequence benchmarks.

Experimental evaluation: SAT NSeq

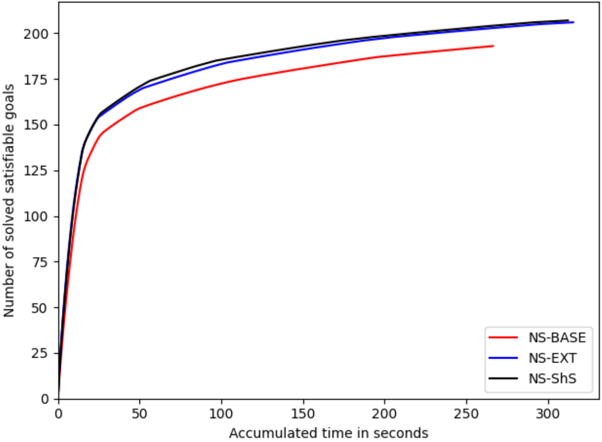

Figure: Number of solved goals by accumulated time (in seconds) on satisfiable quantifier-free *n*-Indexed Sequence benchmarks.

Outline

Conclusion

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Contributions presented in this talk:

- The theory of n-Indexed Sequences.
- Various ways to reason over it.
- Experimental evaluation.

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- Implementation details and formalizations.
- Work on real and integer arithmetic reasoning.
  (Labeled union-find for intervals and difference logic)

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- Add (n-)sequences to Alt-Ergo
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Contributions to software:

- Colibri2
- Alt-Ergo
- Smtml
- Dolmen
- SMT LSP

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Other:

- Co-supervized an intern for 6 months (Félix Loyau-Kahn, Master's student) on using AI for SMT solver selection.

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Publications

- **"On SMT Theory Design: The Case of Sequences"**
  Hichem Rami Ait-El-Hara, François Bobot and Guillaume Bury. LPAR 2024
- **"An SMT Theory for n-Indexed Sequences"**
  Hichem Rami Ait-El-Hara, François Bobot, and Guillaume Bury. SMT 2024
- **"Reasoning over n-indexed sequences in SMT"**
  Hichem Rami Ait-El-Hara, François Bobot, and Guillaume Bury. Acta Informatica 62.3 (Aug. 2025)
- **"Relational Abstractions Based on Labeled Union-Find"**
  Dorian Lesbre, Matthieu Lemerre, Hichem Rami Ait-El-Hara, and François Bobot. PLDI 2025

- **"Constraint Propagation for Bit-Vectors in Alt-Ergo"**
  Hichem Rami Ait-El-Hara, Guillaume Bury, Basile Clément, and Pierre Villemot. SMT 2025

Preprints:

- **"Smt.ml: A Multi-Backend Frontend for SMT Solvers in OCaml"**
  João Madeira Pereira, Filipe Marques, Pedro Adão, Hichem Rami Ait-El-Hara, Léo Andrès, Arthur Carcano, Pierre Chambart, Nuno Santos, and José Fragoso Santos. To be submitted to TACAS 2026.

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Appendix

6. Labeled Union-Find for Arithmetic reasoning

7. NS-BASE and NS-EXT

Reduced Product

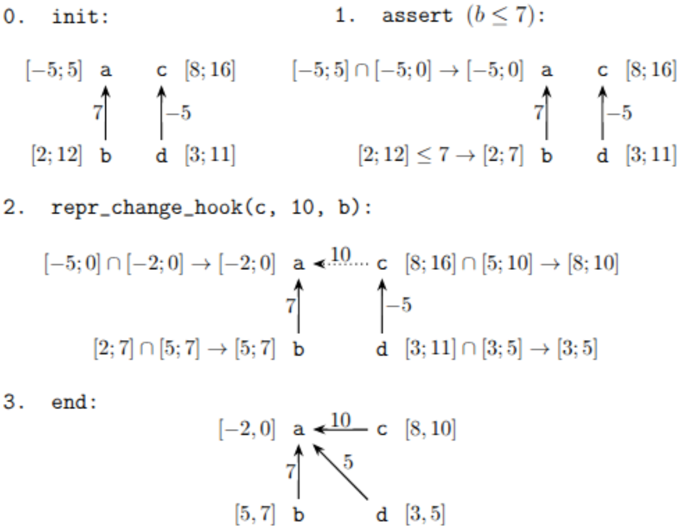

Figure 6.4: Example of the usage of the constant difference relation for constraint propagation over the domain of intervals.

Group Action

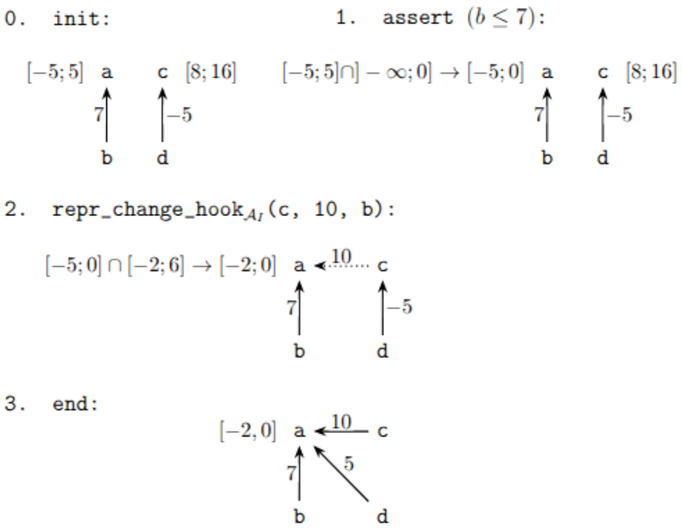

Figure 6.5: Example of the usage of the constant difference relation for constraint factorization over the domain of intervals.

Normal forms

Definition (NSeq term normal form)

For simplicity, we introduce the concatenation operator :: with the invariant:

$$s \mapsto s_1 :: s_2 \implies \mathsf{f}_s = \mathsf{f}_{s_1} \wedge \mathsf{l}_s = \mathsf{l}_{s_2} \wedge \mathsf{f}_{s_2} = \mathsf{l}_{s_1} + 1$$

Normalization

The following rewriting rules are applied whenever possible:

$$\begin{cases} s \mapsto [w_1 ::]x[:: w_2] \\ x \mapsto y :: z \end{cases} \longrightarrow \begin{cases} s \mapsto [w_1 ::]y :: z[:: w_2] \\ x \mapsto y :: z \end{cases}$$

And if $\mathsf{l}_y < \mathsf{f}_y$ is deduced:

$$\begin{cases} s \mapsto [w_1 ::]y :: z[:: w_2] \\ x \mapsto y :: z \end{cases} \longrightarrow \begin{cases} s \mapsto [w_1 ::]z[:: w_2] \\ x \mapsto z \end{cases}$$